Math 246B Lecture 20 Notes

Daniel Raban

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1 Uniqueness of the Γ-Function and Hadamard Factorization of $1/\Gamma$

1.1 Uniqueness of the Γ-function

Last time, we defined the Γ-function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

We saw that $\Gamma \in \mathrm{Hol}(\mathrm{Re}(z) > 0)$ and extends meromorphically to all of $\mathbb{C}$ with simple poles at $\{0, -1, -2, \dots\}$. We also saw that

$$\Gamma(z+1) = z\Gamma(z),$$
$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

the latter of which is called the "reflection identity."

The functional equation actually characterizes Γ.

Proposition 1.1. *Let $f \in \mathrm{Hol}(\mathrm{Re}(z) > 0)$ be such that $f(z+1) = zf(z)$, and assume that f is bounded in $1 \leq \mathrm{Re} \leq 2$. Then $f(z) = f(1)\Gamma(z)$.*

Proof. Consider $\tilde{f}(z) = f(z) - f(1)\Gamma(z)$. We have $\tilde{f}(z+1) = z\tilde{f}(z)$, so $\tilde{f}$ extends meromorphically to $\mathbb{C}$ with simple poles at $\{0, -1, -2, \dots\}$, and we can write

$$\tilde{f}(z) = \frac{\tilde{f}(z+N-1)}{z(z+1)\cdots(z+N)}, \qquad \mathrm{Re}(z) > -N-1.$$

So $\mathrm{Res}(\tilde{f}, -N) = \lim_{z \to -N}(z+N)\tilde{f}(z) = 0$ for all N. So $\tilde{f}$ is entire. Set $\tilde{u}(z) = \tilde{f}(z) = \tilde{f}(z)\tilde{f}(1-z) \in \mathrm{Hol}(\mathbb{C})$, and we get

$$\tilde{u}(z+1) = \tilde{f}(z+1)\tilde{f}(-z) = z\tilde{f}(z)\frac{1}{-z}\tilde{f}(1-z) = -\tilde{u}(z).$$

So $\tilde{u}$ is antiperiodic and bounded in $1 \leq \mathrm{Re}(z) \leq 2$, so $\tilde{u}$ is constant. So we get $\tilde{u}(z) = \tilde{u}(1) = 0$. $\square$

1.2 Hadamard factorization of $1/\Gamma$

Theorem 1.1. *The function $1/\Gamma$ is entire of finite order 1 with the Hadamard factorization*

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{k=1}^{\infty} (1 + z/k) e^{-z/k},$$

where $\gamma = \lim_{N\to\infty} \sum_{n=1}^{N} 1/n - \log(N)$ is the Euler constant.

Proof. We have the reflection identity

$$\frac{1}{\Gamma(z)} = \Gamma(1-z)\frac{\sin(\pi z)}{\pi}$$

for all $z \in \mathbb{C}$. The sine term is of order 1. We have

$$\begin{aligned}\Gamma(z) &= \int_0^1 e^{-t} t^{z-1}\, dt + \int_1^\infty e^{-t} t^{z-1}\, dt \\ &= \sum_{j=0}^{\infty} \int_0^1 \frac{(-t)^j}{j!} t^{z-1}\, dt + \int_1^\infty e^{-t} t^{z-1}\, dt \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+z)} + \underbrace{\int_1^\infty e^{-t} t^{z-1}\, dt}_{\in \mathrm{Hol}(\mathbb{C})}.\end{aligned}$$

The series defines a meromorphic function in $\mathbb{C}$ with poles at $\{0, -1, -2, \dots\}$ since for every compact set $K \subseteq \mathbb{C}$, the functions $(-1)^j/(j!(j+z))$ have no poles in K for $j \geq j_0$ and because $\sum_{j=j_0}^{\infty} (-1)^j/(j!(j+z))$ converges uniformly on K. We get by analytic continuation that

$$\Gamma(1-z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+1-z)} + \int_1^\infty e^{-t} t^{-z}$$

for any z, so

$$\frac{1}{\Gamma(z)} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+1-z)} \frac{\sin(\pi z)}{\pi} + \left(\int_1^\infty e^{-t} t^{-z}\right) \frac{\sin(\pi z)}{\pi}.$$

Now

$$\left|\int_1^\infty e^{-t} t^{-z}\, dt\right| \leq \int_1^\infty e^{-t} e^{|\operatorname{Re}(z)|}\, dt$$

Let $|\operatorname{Re}(z)| \leq n < 1 + |\operatorname{Re}(z)|$, where $n \in \mathbb{N}$.

$$\leq n!$$

$$
\begin{aligned}
&\leq n^n \\
&\leq e^{(1+|z|)\log(1+|z|)},
\end{aligned}
$$

so we get

$$\left| \left(\int_1^\infty e^{-t} t^{-z} \right) \frac{\sin(\pi z)}{\pi} \right| \leq C e^{C(1+|z|)\log(1+|z|)}.$$

If $|\operatorname{Im}(z)| \geq 1$, then

$$\left| \left(\sum_{j=0}^\infty \frac{(-1)^j}{j!(j+1-z)} \right) \frac{\sin(\pi z)}{\pi} \right| \leq C e^{\pi |z|}.$$

The same estimate holds if $\operatorname{Re}(z) \leq 1/2$. Let $k \in \mathbb{N}_+$ with $k \geq 1$ be such that $k - 1/2 \leq \operatorname{Re}(z) < k + 1/2$. Then

$$\left(\sum_{j=0}^\infty \frac{(-1)^j}{j!(j+1-z)} \right) \frac{\sin(\pi z)}{\pi} = \underbrace{\frac{(-1)^k}{k!(k-z)} \frac{\sin(\pi z)}{\pi}}_{O(1)} + O(1) e^{\pi |z|}.$$

It follows that the order of $1/\Gamma$ is ≤ 1.

To see that the order is ≥ 1, write

$$\Gamma(z) = \frac{\Gamma(z+N+1}{z(z+1)\cdots(z+N)}, \qquad \operatorname{Re}(z) > -N-1.$$

and take $z = N - 1/2$. Then

$$\left| \frac{1}{\Gamma(-N-1/2)} \right| \geq \frac{1}{N!} \geq \frac{1}{C} N^N e^{-N}$$

by Stirling's formula. So the order of $1/\Gamma$ is exactly 1.

By Hadamard's theorem, we get

$$\frac{1}{\Gamma(z)} = e^{\alpha z + \beta} z \prod_{k=1}^\infty (1 - z/k) e^{-z/k}.$$

Multiply both sides by $\Gamma(z)$, and let $z \to 0$. We get

$$1 = \lim_{z \to 0} e^{\alpha z + \beta} \Gamma(z) z = e^\beta,$$

so $\beta = 0$. To compute $\alpha \in \mathbb{R}$, take $z = 1$ in the expression for $1/\Gamma$:

$$1 = \frac{1}{\Gamma(z)} e^\alpha \prod_{k=1}^\infty (1 + 1/k) e^{-1/k},$$

so

$$e^{-\alpha} = \lim_{N\to\infty} \exp\left(-\sum_{k=1}^{N} 1/k + \sum_{k=1}^{N} \log(k+1) - \log(k) \right).$$

We get that

$$\alpha = \lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{K} - \log(N). \qquad \square$$

Next, we will discuss the range of holomorphic functions with Picard's theorems.

Theorem 1.2 (Picard's little theorem)**.** *Let* $f \in \mathrm{Hol}(\mathbb{C})$ *be entire and nonconstant. Then the range* $f(\mathbb{C})$ *omits at most 1 point of* $\mathbb{C}$*.*